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# On a broad class of univalent functions

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## Abstract

The purpose of the present paper is to give some univalence conditions for a broad class of analytic functions. Moreover, we consider some special cases as corollaries of the main results.

**2000 Mathematics Subject Classification.** 30C45.

**Key Words and Phrases.** univalent function, convex function, starlike function, close-to convex function, Bazilevič function of type  $\beta$  and  $\phi$ -like function.

## 1. Introduction

It is well known that if  $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$  is analytic in  $D = \{z \mid |z| < 1\}$  and we suppose that  $f(z)$  satisfies one of the following conditions

$$\operatorname{Re} f'(z) > 0 \quad \text{in } D \quad (1.1)$$

$$1 + \operatorname{Re} \frac{zf''(z)}{f'(z)} > 0 \quad \text{in } D, \quad (1.2)$$

$$\operatorname{Re} \frac{zf'(z)}{f(z)} > 0 \quad \text{in } D, \quad (1.3)$$

$$\operatorname{Re} \frac{zf'(z)}{g(z)} > 0 \quad \text{in } D, \quad (1.4)$$

$$\operatorname{Re} \frac{zf'(z)}{\phi(f(z))} > 0 \quad \text{in } D, \quad (1.6)$$

where  $g(z) = z + \sum_{n=2}^{\infty} b_n z^n$  is analytic and satisfies the condition

$$\operatorname{Re} \frac{zg'(z)}{g(z)} > 0 \quad \text{in } D$$

or  $g(z)$  is starlike in  $D$ ,  $0 < \beta$  and  $\phi$  is analytic on  $f(D)$  with  $\phi(0) = 0$  and  $\operatorname{Re} \phi'(0) > 0$ , then  $f(z)$  is univalent in  $D$  and we call  $f(z)$  when  $f(z)$  satisfies the condition (1.1), (1.2), (1.3), (1.4), (1.5) and (1.6) as a Noshiro-Warschawski function, a convex function, a starlike function, a close-to convex function, a Bazilevič function of type  $\beta$  and  $\phi$ -like function, respectively.

It is the propose of the present paper to introduce a broad class of analytic functions and to investigate some sufficient conditions for univalence of the class.

## 2. Main Results

**Theorem 1.** Let  $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$  be analytic in  $D$  and suppose that

$$\operatorname{Re} \frac{zf'(z)}{\varphi(f(z), z)} > 0 \quad \text{in } D,$$

where  $\varphi(f(z), z)$  is analytic in  $(f(D), D)$  and

$$\frac{d \arg \varphi(w, re^{i\theta})}{d\theta} > 0 \quad \text{in } (f(D), D)$$

for  $z = re^{i\theta}$ ,  $0 < r < 1$  and  $0 \leq \theta < 2\pi$ . Then  $f(z)$  is univalent in  $D$ .

*Proof.* If there exists a point  $z_0$ ,  $|z_0| < 1$  such that

$$f(z) \text{ is univalent for } |z| < |z_0|$$

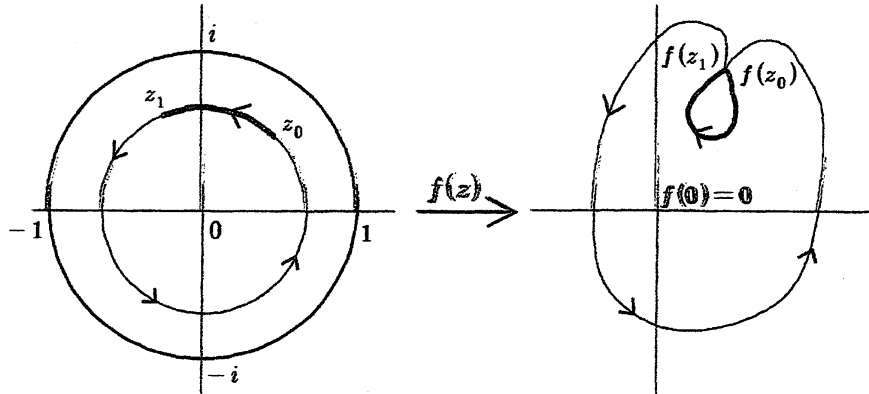
and

$$f(z) \text{ is not univalent for } |z| \leq |z_0|,$$

then there exists a point  $z_1$ ,  $z_0 \neq z_1$ ,  $|z_0| = |z_1|$ ,  $z_0 = |z_0|e^{i\theta_0}$ ,  $z_1 = |z_0|e^{i\theta_1}$  and  $0 \leq \theta_0 < \theta_1 < 2\pi$  for which

$$f(z_0) = f(z_1),$$

as we see in the following figures.



Let

$$C_z = \{z \mid |z| = |z_0|, z = |z_0|e^{i\theta} \text{ and } \theta_0 \leq \theta \leq \theta_1\}.$$

Then from the hypothesis, we have

$$\begin{aligned} -\pi &= \int_{C_z} d \arg df(z) = \int_{C_z} d \arg \frac{df(z)}{dz} dz \\ &= \int_{C_z} d \arg \left( \frac{zf'(z)}{\varphi(f(z), z)} \right) + \int_{C_z} d \arg \left( \frac{dz}{z} \right) + \int_{C_z} d \arg \varphi(f(z), z) \\ &> -\pi + (\arg \varphi(f(z_1), z_1) - \arg \varphi(f(z_0), z_0)) \\ &= -\pi + (\arg \varphi(f(z_0), z_1) - \arg \varphi(f(z_0), z_0)) \\ &= -\pi + \int_{\theta_0}^{\theta_1} \frac{d \arg \varphi(f(z_0), |z_0|e^{i\theta})}{d\theta} d\theta \\ &> -\pi. \end{aligned}$$

This is contradiction and so, we completes the proof. □

**Corollary 1.** Let  $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$  be analytic in  $D$  and suppose that

$$\operatorname{Re} \frac{zf'(z)}{\varphi(f(z), z)} > 0 \quad \text{in } D,$$

where

$$\frac{d \arg \varphi(w, re^{i\theta})}{d\theta} > 0 \quad \text{in } (f(D), D)$$

and  $\varphi(f(z), z)$  satisfies one of the following conditions

$$\varphi(f(z), z) = z = re^{i\theta} \quad [5, 8], \quad (2.1)$$

$$\varphi(f(z), z) = zf'(z) = re^{i\theta} f'(e^{i\theta}) \quad [7], \quad (2.2)$$

$$\varphi(f(z), z) = f(z) = f(re^{i\theta}) \quad [4], \quad (2.3)$$

$$\varphi(f(z), z) = g(z) = g(re^{i\theta}) \quad [2, 3, 6], \quad (2.4)$$

$$\varphi(f(z), z) = \varphi(w, re^{i\theta}) = w^{1-\beta} g(z)^\beta = w^{1-\beta} g(re^{i\theta}) \quad [1], \quad (2.5)$$

where  $z = re^{i\theta}$ ,  $0 \leq r < 1$ ,  $0 \leq \theta < 2\pi$ ,  $0 < \beta$  and  $g(z) = z + \sum_{n=2}^{\infty} b_n z^n$  is analytic and starlike in  $D$  or

$$\operatorname{Re} \frac{zg'(z)}{g(z)} > 0 \quad \text{in } D.$$

Then  $f(z)$  is univalent in  $D$ .

*Proof.* For the case (2.1), from hypothesis we have

$$\operatorname{Re} \frac{zf'(z)}{\varphi(f(z), z)} = \operatorname{Re} \frac{zf'(z)}{z} = \operatorname{Re} f'(z) > 0 \quad \text{in } D$$

and

$$\frac{d \arg \varphi(w, re^{i\theta})}{d\theta} = \frac{d\theta}{d\theta} = 1 > 0 \quad \text{in } D.$$

Applying Theorem 1,  $f(z)$  is univalent in  $D$ .

For the case (2.2), we have

$$\operatorname{Re} \frac{zf'(z)}{\varphi(f(z), z)} = \operatorname{Re} \frac{zf'(z)}{zf'(z)} = 1 > 0 \quad \text{in } D$$

and

$$\begin{aligned} \frac{d \arg \varphi(w, re^{i\theta})}{d\theta} &= \frac{d \arg zf'(z)}{d\theta} = \frac{d \arg(\frac{z}{dz})}{d\theta} + \frac{d \arg df(z)}{d\theta} \\ &= \frac{d \arg df(z)}{d\theta} = 1 + \operatorname{Re} \frac{zf''(z)}{f'(z)} > 0 \quad \text{in } D. \end{aligned}$$

This shows that  $f(z)$  is convex and univalent in  $D$ .

For the case (2.3), we have

$$\operatorname{Re} \frac{zf'(z)}{\varphi(f(z), z)} = \operatorname{Re} \frac{zf'(z)}{f(z)} > 0 \quad \text{in } D$$

and

$$\frac{d \arg \varphi(w, re^{i\theta})}{d\theta} = \frac{d \arg f(z)}{d\theta} = \operatorname{Re} \frac{zf'(z)}{f(z)} > 0 \quad \text{in } D.$$

This shows that  $f(z)$  is starlike in  $D$ .

For the case (2.4), we have

$$\operatorname{Re} \frac{zf'(z)}{\varphi(f(z), z)} = \operatorname{Re} \frac{zf'(z)}{g(z)} > 0 \quad \text{in } D$$

and

$$\frac{d \arg \varphi(w, re^{i\theta})}{d\theta} = \frac{d \arg g(z)}{d\theta} = \operatorname{Re} \frac{zg'(z)}{g(z)} > 0 \quad \text{in } D.$$

This shows that  $f(z)$  is univalent in  $D$  and close-to-convex in  $D$ .

For the case (2.5), we have

$$\operatorname{Re} \frac{zf'(z)}{\varphi(f(z), z)} = \operatorname{Re} \frac{zf'(z)}{f(z)^{1-\beta}g(z)^\beta} > 0 \quad \text{in } D$$

and

$$\frac{d \arg \varphi(w, re^{i\theta})}{d\theta} = \frac{d \arg w^{1-\beta}g(re^{i\theta})^\beta}{d\theta} = \beta \frac{d \arg g(re^{i\theta})}{d\theta} = \beta \operatorname{Re} \frac{zg'(z)}{g(z)} > 0 \quad \text{in } D$$

where  $0 < \beta$ . This shows that  $f(z)$  is univalent in  $D$  and  $f(z)$  is Bazilevič function of type  $0 < \beta$ .  $\square$

If  $f(z)$  is a Bazilevič function of type  $\beta$ , then  $\beta$  must be a positive real number. But we can obtain the following theorem.

**Theorem 2.** Let  $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$  be analytic in  $D$  and suppose that

$$\left| \arg \frac{zf'(z)}{f(z)^{1-\beta}g(z)^\beta} \right| < \frac{\pi}{2} \alpha \quad \text{in } D,$$

where  $0 < \alpha < 1$ ,  $\beta < 0$ ,  $\alpha - 2\beta < 1$  and  $g(z) = z + \sum_{n=2}^{\infty} b_n z^n$  is analytic and starlike in  $D$ . Then  $f(z)$  is univalent in  $D$ .

*Proof.* If there exists a point  $z_0$ ,  $|z_0| < 1$  such that

$$f(z) \text{ is univalent for } |z| < |z_0|$$

and

$$f(z) \text{ is not univalent for } |z| \leq |z_0|,$$

then there exists a point  $z_1$ ,  $z_0 \neq z_1$ ,  $|z_1| = |z_0|$ ,  $z_0 = |z_0|e^{i\theta_0}$ ,  $z_1 = |z_0|e^{i\theta_1}$  and  $0 \leq \theta_0 < \theta_1 < 2\pi$  for which

$$f(z_0) = f(z_1).$$

Then the image picture under the mapping  $w = f(z)$  for  $|z| = |z_0|$  is the same as the picture of the proof of Theorem 1.

Let

$$C_z = \{z \mid |z| = |z_0|, z = |z_0|e^{i\theta} \text{ and } \theta_0 \leq \theta \leq \theta_1\},$$

$$C'_z = \{z \mid |z| = |z_0|\} - C_z,$$

and

$$\Gamma'_w = f(C'_z).$$

Then we have

$$\begin{aligned} 3\pi &= \int_{\Gamma'_w} d \arg dw = \int_{\Gamma'_w} d \arg df(z) \\ &= \int_{C'_z} d \arg \left( \frac{zf'(z)}{f(z)^{1-\beta}g(z)^\beta} \right) + \int_{C'_z} d \arg \left( \frac{dz}{z} \right) + \int_{C'_z} d \arg f(z)^{1-\beta} + \int_{C'_z} d \arg g(z)^\beta \\ &= \int_{C'_z} d \arg \left( \frac{zf'(z)}{f(z)^{1-\beta}g(z)^\beta} \right) + (1-\beta) \int_{C'_z} d \arg f(z) + \beta \int_{C'_z} d \arg g(z) \\ &< \alpha\pi + (1-\beta)2\pi = \pi(2+\alpha-2\beta) < 3\pi. \end{aligned}$$

This is a contradiction and so, we completes the proof.  $\square$

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